Fragility and robustness of the Kardar-Parisi-Zhang universality class

Matteo Nicoli,[1] Rodolfo Cuerno,[2] and Mario Castro[3]

[1] *Laboratoire de Physico-Chimie des Polymères et Milieux Dispersés (PPMD), ESPCI ParisTech, CNRS, UPMC, 10 rue Vauquelin, 75231 Paris Cédex 05, France*

[2] *Departamento de Matemáticas and Grupo Interdisciplinar de Sistemas Complejos (GISC), Universidad Carlos III de Madrid, Avenida de la Universidad 30, E-28911 Leganés, Spain*

[3] *GISC and Grupo de Dinámica No Lineal (DNL), Escuela Técnica Superior de Ingeniería (ICAI), Universidad Pontificia Comillas, E-28015 Madrid, Spain*

(Dated: April 23, 2013)

We assess the dependence on substrate dimensionality of the asymptotic scaling behavior of a whole family of equations that feature the basic symmetries of the Kardar-Parisi-Zhang (KPZ) equation. Even for cases in which, as expected from universality arguments, these models display KPZ values for the critical exponents and limit distributions, their behavior deviates from KPZ scaling for increasing system dimensions. Such a fragility of KPZ universality contradicts naive expectations, and questions straightforward application of universality principles for the continuum description of experimental systems. Still, we find that the ensuing limit distributions do coincide with those of the KPZ class in one and two dimensions, demonstrating the robustness of the latter under changes of the critical exponent values.

PACS numbers: 05.10.Gg, 68.35.Ct, 81.15.Aa, 05.45.-a,

One of the most powerful concepts in contemporary Statistical Mechanics is the idea of universality, by which microscopically dissimilar systems show the same large scale behavior, provided they are controlled by interactions that share dimensionality, symmetries, and conservation laws. Being rooted in the behavior of equilibrium critical systems [1], universality has more recently allowed to describe scaling behavior in far-from-equilibrium phenomena [2–4], such as e.g. stock market fluctuations [5], crackling-noise [6], or random networks [7]. In complex systems like these, universality provides an enormously simplifying framework, as significant descriptions can be put forward on the basis of the general principles just mentioned.

Among non-equilibrium systems, particularly conspicuous cases are those displaying generic scale invariance, for which scaling behavior occurs throughout parameter space [8]. Examples include self-organized-critical [9] and driven-diffusive systems [10], or surface kinetic roughening [11]. Actually, a paradigmatic model for rough interfaces, the Kardar-Parisi-Zhang (KPZ) equation [12]

$$\partial_t h = \nu \nabla^2 h + \frac{\lambda}{2} (\nabla h)^2 + \eta(\mathbf{x}, t), \qquad (1)$$

is very recently proving itself as a remarkable instance of universal behavior. Here, $h(\mathbf{x}, t)$ is a height field above substrate position $\mathbf{x} \in \mathbb{R}^d$ at time t, and η is Gaussian white noise with zero mean and variance $2D$. Thus, the exact asymptotic height distribution function has been very recently obtained for $d = 1$ [13, 14]: it is given by the largest-eigenvalue distribution of large random matrices in the Gaussian unitary (GUE) (orthogonal, GOE) ensemble, the Tracy-Widom (TW) distribution, for globally curved (flat) interfaces, as proposed in [15], see reviews in [16, 17]. Besides elucidating fascinating connections with probabilistic and exactly solvable systems, these and related results are allowing to show that, not only are the critical exponent values common to members of this universality class, but also the distribution functions and limiting processes are shared by discrete models and continuum equations [18], and by diverse experimental systems, from turbulent liquid crystals [19] to drying colloidal suspensions [20].

In view of the success for one-dimensional (1D) substrates, a natural important step is to assess the behavior of the KPZ universality class when changing space dimension, analogous to e.g. the experimental change from 2D to 1D behavior for ferromagnetic nanowires, that nonetheless occurs within the creeping-domain-wall class [21]. Thus, working with discrete models and the continuum equation itself, it has been very recently found [22] that, for $d = 2$, indeed universal distributions also control fluctuation phenomena in the KPZ class, providing higher-dimensional analogs of the TW distributions. Again this makes a strong statement on universality, beyond critical exponent values that were already known to be shared by the KPZ equation, many discrete models [23], and some experiments [24], although much less than expected [25]. This fact calls for further experimental verification, akin to that recently provided [19, 20] for the one-dimensional case.

In this work we report a fragility of the KPZ universality class with respect to space dimension. Namely, we show that a family of continuum equations that share the same symmetries and conservation laws of the KPZ equation, and are accurately described by its asymptotic distribution function in $d = 1$, do *not* share with it even the values of the scaling exponents in $d = 2$. Analogous behavior had been found earlier for discrete models of (conservative) surface growth [26], namely, a change of

the universality class of a given system with d. Here we show that it takes place also for non-conserved dynamics, and at the level of continuum equations. Note, this is not a change in the universality class as a response to changes in appropriate system parameters for a fixed dimension, as seen e.g. in the context of Barkhausen criticality [27]. The lack of universality that we find points to a serious difficulty in the identification of the appropriate universality class for experimental systems, as it prevents cursory use of universality arguments to propose theoretical descriptions, stressing the need for physically motivated models [25]. This fact should be borne in mind, specially in view of the timely interest of experimental validation for KPZ universality in the 2D case.

We consider the following equation [28]

$$\partial_t h_{\mathbf{k}}(t) = (\nu k^{\mu} - \mathcal{K} k^2) h_{\mathbf{k}}(t) + \frac{\lambda}{2}\mathcal{F}[(\nabla h)^2]_{\mathbf{k}} + \eta_{\mathbf{k}}(t), \quad (2)$$

where $k = |\mathbf{k}|$, and $\nu, \mathcal{K} > 0$, $\mathcal{F}$ is space Fourier transform, and $h_{\mathbf{k}}(t)$ and $\eta_{\mathbf{k}}(t)$ are the $\mathbf{k}$-th modes of the height and of the noise fields, respectively. Equation (2) perturbs the KPZ equation (1) through the introduction of the linear term with coefficient ν, where $0 < \mu < 2$. This family of equations includes celebrated systems, such as e.g. the Kuramoto-Sivashinsky (KS) (take $\mu \to 2$ and replace k^2 with k^4) and the Michelson-Sivashinsky ($\mu = 1$) equations [28], that combine pattern formation at short time and length scales with asymptotic kinetic roughening [29]. For specific μ values, Eq. (2) actually describes *quantitatively* surface growth experiments of diffusion-limited processes, such as plasma etching [30], electrochemical [31], and chemical vapor [32] deposition.

Equation (2) complies with the standard symmetries of the KPZ class. Namely, it corresponds to non-conserved dynamics, is isotropic and reflection invariant in the substrate coordinates $\mathbf{x}$, invariant under arbitrary shifts $h \to h + \text{const.}$, breaks the up-down symmetry $h \leftrightarrow -h$, and satisfies Galilean invariance [33]. Two additional features are to be noted, namely, the non-analytic dependence [34] of the linear terms on k, and the morphological instability [25]. The former induces *non-locality* of the equation when written in real space [35, 36], and the latter breaks scale invariance at short time and length scales, which is restored back at large scales along the dynamics, as in the KS system [11]. Indeed, as borne out by numerical [28] and dynamic renormalization group [33] results, the asymptotic behavior of Eq. (2) fulfills the Family-Vicsek (FV) scaling ansatz [11], implying that the surface structure factor or power spectral density, $S(k,t) = \langle |h_{\mathbf{k}}(t)|^2 \rangle$, scales at long times as $S(k, t \to \infty) \sim 1/k^{2\alpha+d}$, with a well-defined value of the roughness exponent. The crossover wave-vector value separating white noise from correlated behavior also shows FV behavior as $k_c \sim t^{-1/z}$, leading to scaling for the global roughness $W(t)$ (root mean square fluctuation of the surface height) with time as $W \sim t^{\beta}$, with

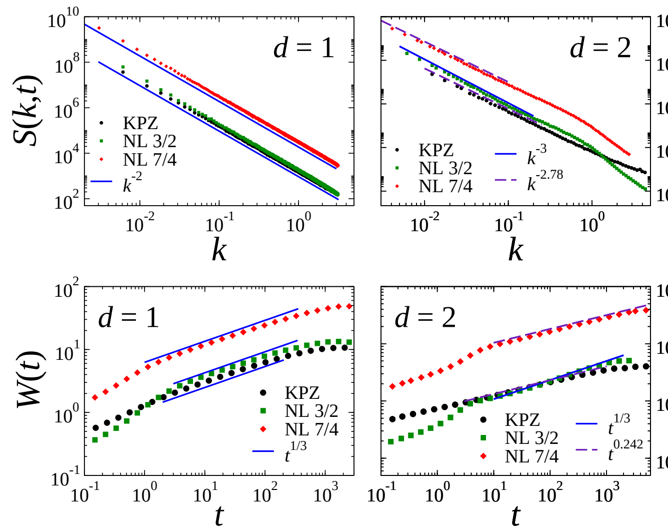

FIG. 1: (Color online). Numerical simulations of Eq. (1) (●) and Eq. (2) for $\mu = 3/2$ (■) and $\mu = 7/4$ (♦). Surface structure factor (upper row) and roughness (lower row) for $d = 1$ (left column) and $d = 2$ (right column). Solid and dashed lines represent power-law behaviors as indicated, which are discussed in the main text. All the observables have been averaged over 10^3 (10^2) different noise realizations for $d = 1$ ($d = 2$), starting from a flat initial condition. Error bar are smaller than symbol sizes. All units are arbitrary.

$\beta = \alpha/z$. The values of these critical exponents depend on μ, and correspond to the KPZ universality class in d dimensions, provided $z_{\rm KPZ}(d) \leq \mu < 2$, where $z_{\rm KPZ}(d)$ is the KPZ value of the dynamic exponent for the corresponding dimension. Note that, for the morphologically unstable condition $\nu > 0$ that we consider, the nonlinearity is dynamically relevant for any value of μ (even if it may not control scaling behavior), while for the morphologically stable situation ($\nu < 0$) scaling in Eq. (2) is controlled by the linear terms for small μ values [28, 33, 37].

In Fig. 1 we show numerical simulations of Eq. (2) for $\mu = 3/2, 7/4$ (we denote these by NL, for non-local) and, as a reference, for the KPZ equation itself, for both $d = 1, 2$, using a pseudospectral scheme as in [28, 31] and parameters reported in Table I. Both values of μ are larger than or equal to $z_{\rm KPZ}(1) = 1/2$, thus for $d = 1$ the behavior is well described by KPZ exponents $\beta_{\rm KPZ}(1) = 1/3$ and $\alpha_{\rm KPZ}(1) = 1/2$.

Universal behavior here *goes beyond* exponent values. Thus, using the Ansatz [15]

$$h \simeq v_\infty t + \mathrm{sgn}(\lambda)(\Gamma t)^{\beta}\chi, \quad (3)$$

we can measure the fluctuations of the interface around its mean value $v_\infty t$, i.e. $\chi = \mathrm{sgn}(\lambda)(h - v_\infty t)/(\Gamma t)^{\beta}$. Through Γ we normalize the variance of this stochastic process to the variance of the TW-GOE distribution (that is 0.638) and we are able to compare them. For the estimation of v_∞ and Γ we followed the procedure

Equation		$\mathcal{K}$	λ	D	L	Δt	v_∞	Γ	t^*
	KPZ		5.0	1.46	1024	0.002	3.6990	5.620	1000
$d=1$	NL 3/2	1.0	5.0	0.50	1024	0.001	4.2675	11.95	500
	NL 7/4	1.7	2.5	12.5	2048	0.002	18.4625	310.0	500
	KPZ		2.5	1.46	512	0.01	2.1307	0.596	250
$d=2$	NL 3/2	0.5	10	0.50	1024	0.01	1.4790	0.212	250
	NL 7/4	1.0	1.0	50.0	1536	0.02	8.7425	2725	250

TABLE I: Parameters used for the numerical integrations reported in this work. NL stands for the non-local models, i.e. Eq. (2) with μ equal to 3/2 or 7/4. In all cases $\nu = 1$. L is the size of the 1D domain, or the edge of the 2D square, used for simulations in Fig. 1. t^* is the time used in the computation of the quantities reported in Figs. 2 and 3. See more details in [38]. All units are arbitrary.

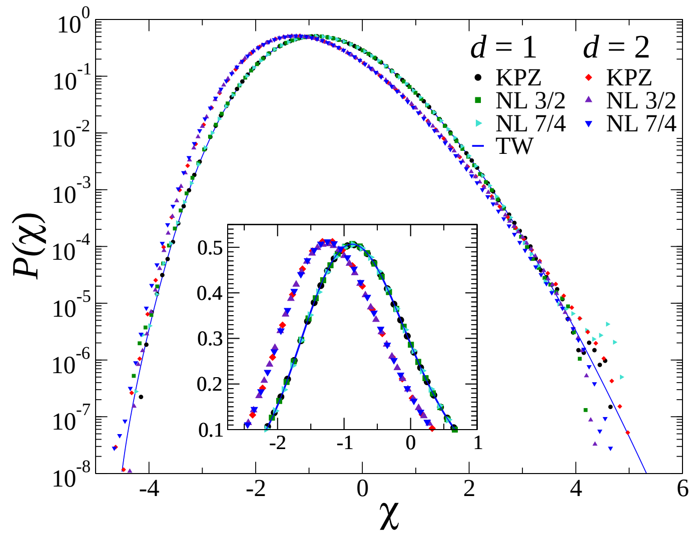

FIG. 2: (Color online). Height distributions for Eq. (1) (●, $d = 1$; ♦, $d = 2$) and Eq. (2) for $\mu = 3/2$ (■, $d = 1$; ▲, $d = 2$) and $\mu = 7/4$ (►, $d = 1$; ▼, $d = 2$). The variable χ is defined in the text. The solid blue line is the TW (GOE) distribution expected for $d = 1$ [39]. For 1D (2D) simulations $P(\chi)$ is estimated from 2048 (1024) independent runs. Inset: zoom of main panel, in linear representation. All units are arbitrary.

described in [18]. Values for these constants are reported in Table I (for more details, see [38]). As clearly shown in Fig. 2, the random variable χ is time-independent and distributed according to the GOE TW distribution for the three equations. The solid line in Fig. 2 has been calculated from the solution of the Painlevé II differential equation [39], and normalized according to [15].

Further universal behavior is seen to occur, as in other instances of the 1D KPZ class [18], for the two-point correlation function $C(x,t) = \langle h(x_0 + x, t)h(x_0, t)\rangle - \langle h\rangle^2$, that scales as $C(x,t) \sim (2\Gamma t)^{2/3} g_1(u)$, where we consider $u = (Ax/2)/(2\Gamma t)^{2/3}$ with $g_1(u)$ the covariance of the Airy_1 process [18, 40], and $A = (2\Gamma/\lambda)^{1/2}$ for continuum equations (for discrete models, A is estimated from the local roughness [18]). As suggested by results in Fig. 2, even for the 1D non-local models considered here, the rescaled two-point correlation function collapses perfectly onto $g_1(u)$, see Fig. 3.

However, when we increase the system dimension to $d = 2$, a remarkable departure from KPZ scaling occurs that depends on the relative values of μ and $z_{\rm KPZ}(2) \approx 1.61$. Thus, while Eq. (2) is still well described by KPZ exponent values for "large" $\mu = 7/4$ as deduced from Fig. 1, namely, $\alpha \simeq 0.39$ (compare $\alpha_{\rm KPZ}(2) \approx 0.39$ [23]) and $z \simeq 1.61$, the "small" $\mu = 1/2$ system has the same exponent values as for $d = 1$.

This fact bears important consequences on the continuum modeling of systems, in particular of an experimental type, that are presumably in the KPZ universality class. Take the 1D case as an example. Eq. (2) having the same symmetries as the KPZ equation, one might postulate the latter as a model description for a given experiment. But suppose the actual physical interactions lead to the occurrence of morphological instabilities (as it happens only too often in surface growth experiments [25]), in such a way that a better description is provided by Eq. (2) for $\mu = 1/2$. This will not change the 1D scaling behavior with respect to KPZ universality, even at the level of height distributions or correlation functions. However, if one were able to perform an experiment for the 2D generalization of the system, a departure from KPZ behavior would be obtained, with the conclusion that the universality class of the physical system would *not* be KPZ. One might argue that increasing d for a fixed μ makes interactions more non-local in real space [35], and that the present fragility of KPZ scaling is only superficial [4]. However, this does not circumvent the need, for a given physical system, to assess in detail the occurrence of e.g. morphological instabilities and/or the range of interactions, in order to argue for the correct universality class on a safe basis. In any case, this requires going beyond symmetry principles to provide the sought-for continuum description. Note that, starting out with a higher value of μ that leads to KPZ scaling both in $d = 1$ and 2, such as $\mu = 7/4$, only pushes departure from KPZ scaling up to a higher dimension $d_{7/4}$, such that $z_{\rm KPZ}(d_{7/4}) > 7/4$, which will occur below the upper (if finite) critical dimension d_c for the KPZ universality class, at which $z_{\rm KPZ}(d_c) = 2$.

As a bonus from the above analysis, we obtain, on the contrary, an unexpected *robustness* of the KPZ universal height distribution $P(\chi)$ for $d = 2$, see Fig. 2. Namely, we obtain that the shape of $P(\chi)$ is robust to changes in scaling exponent values, e.g. between those of the KPZ equation and those of Eq. (2) for $\mu = 1/2$ and $d = 2$. Thus, $P(\chi)$ turns out to be shared by systems that belong to different universality classes, as defined by exponent values. We can note that all the equations studied in this work share the same so-called Galilean exponent relation,

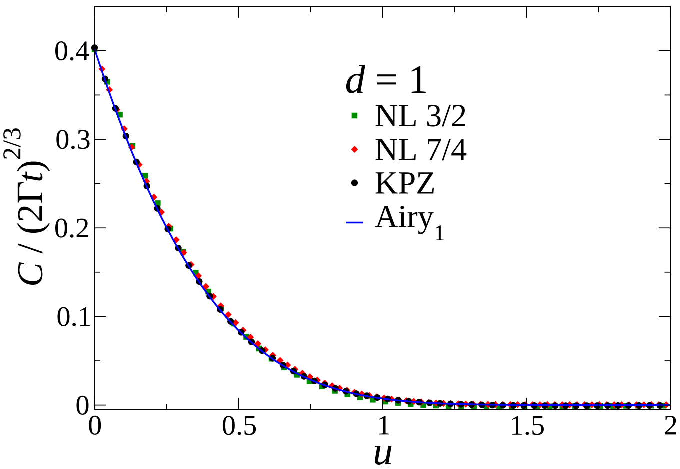

FIG. 3: (Color online). 1D height-height correlation function for Eq. (1) (●) and Eq. (2) for $\mu = 3/2$ (■) and $\mu = 7/4$ (♦). Here, $u = x\sqrt{\Gamma/2\lambda}/(2\Gamma t^*)^{2/3}$ while $C(x, t = t^*)$ is measured from the same surfaces employed to estimate $P(\chi)$ in Fig. 2. The solid blue line provides the covariance of the Airy$_1$ process [40]. All units are arbitrary.

$\alpha + z = 2$, which characterizes, not only the KPZ fixed point [11], but also the fixed point that governs the low$-\mu$ "phase" of Eq. (2) [28, 33]. This scaling relation is quite general and occurs in many stochastic systems, as e.g. for first-passage percolation and related problems [41].

In conclusion, we have found a fragility of the KPZ universality class with respect to space dimension, when perturbed by morphological instabilities combined with non-local interactions, within the experimentally substantiated family of equations, Eq. (2). Note, an important perturbation of the KPZ equation by instabilities that also respects its space symmetries, is also known to occur in the celebrated (noisy) KS system, which is a local equation known to lead to KPZ scaling, both in the 1D [11] and 2D [42] cases. We recall that earlier results have also suggested non-universal behavior for the KPZ class in $d > 1$. E.g. for increasing d, details of the noise distribution have been reported to become relevant to the dynamics [43]. Important issues remain indeed open with respect to the dimensional behavior of the KPZ equation and class, like the existence and value of an upper critical dimension [44], or even making mathematical sense of solutions for $d > 1$ [17].

Although non-equilibrium universality classes are frequently expected to be more fragile than equilibrium ones [2], a highly non-trivial question is to identify, if existent, the type of perturbation that is taking place here and assess its actual importance [4]. In our case, even if the present fragility might be questioned in view of the non-local nature of the perturbation, its occurrence is not easily circumvented by the symmetry arguments that are usually in use for the theoretical description of kinetic roughening phenomena. This seems an important *caveat*, especially in view of the current quest for 2D KPZ scaling behavior in experimental systems. As implied by our results, in order to assign a universality class to a given system, one would need to explore its behavior under a change in d. However, for many experimental systems modifying the space dimension may be hard to achieve without significantly altering the basic interactions that take place. For instance, basic properties of fluid flow can drastically change from a quasi-2D Hele-Shaw cell to a 3D system, while keeping all additional conditions unchanged [45]. This stresses the need for detailed modeling of the specific peculiarities of the system under study, undermining the promise of universality as the main toolbox for kinetically rough systems. An analogous situation occurs in the context of pattern formation, where Goldstone modes associated with the shift symmetry $h \to h +$ const. prevent the existence of a universal amplitude equation [46]. In such contexts, modeling has to be done on a system-specific basis. We note that in these cases symmetry arguments can be enhanced by multiple scales approaches in order to put forward general continuum models that successfully describe experimental systems [47]. One can ponder [34] whether analogous generalized approaches would be successful in the presence of non-localities and noise.

We have also found that, somewhat unexpectedly, the height distributions associated with KPZ universality in one and two dimensions are more general than this very same universality class, also applying to systems with different scaling exponents, but for which the Galilean relation $\alpha + z = 2$ holds. This seems reminiscent of the occurrence of the same probability distribution for global quantities characterizing many equilibrium and non-equilibrium strongly-correlated systems [48]. A similar analogy has been recently noticed in the 2D KPZ context from a different perspective [49]. On the other hand, the TW distribution is known to occur for many other disordered systems, like spin glasses [50]. Further inquiry into these issues seems a promising route to improve our understanding of the role of dimensionality in the context of the KPZ equation and universality class, and for systems with generic scale invariance at large.

We thank E. Moro, J. Rodríguez-Laguna, S. N. Santalla, and P. Vivo for discussions, and F. Bornemann for providing the covariance of the Airy$_1$ process. This work has been partially supported through grants FIS2009-12964-C05-01 and FIS2009-12964-C05-03 (MICINN, Spain), and FIS2012-38866-C05-01 (MEC, Spain).

[1] K. G. Wilson and J. Kogut, Phys. Rep. **12**, 75 (1974).
[2] J. Marro and R. Dickman, *Nonequilibrium Phase Transitions in Lattice Models* (Cambridge University Press,

Cambridge, UK, 1999).
[3] G. Ódor, Rev. Mod. Phys. **76**, 663 (2004).
[4] M. A. Muñoz, AIP Conf. Proc. **1332**, 111 (2011).
[5] *Complex Systems in Finance and Econometrics*, edited by R. A. Meyers (Springer, New York, 2011).
[6] J. P. Sethna, K. A. Dahmen, and C. R. Myers, Nature (London) **419**, 242 (2001).
[7] R. Albert and A.-L. Barabási, Rev. Mod. Phys. **74**, 47 (2002).
[8] D. Belitz, T. R. Kirkpatrick, and T. Vojta, Rev. Mod. Phys. **77**, 579 (2005).
[9] G. Grinstein, in *Scale Invariance, Interfaces, and Non-Equilibrium Dynamics*, edited by A. J. McKane, M. Droz, J. Vannimenus, and D. Wolf (Plenum Press, New York, 1995).
[10] B. Schmittmann and R. K. P. Zia, in *Phase transitions and critical phenomena* **17**, edited by C. Domb and J. L. Lebowitz (Academic Press, London, 2000).
[11] J. Krug, Adv. Phys. **46**, 139 (1997).
[12] M. Kardar, G. Parisi, and Y.-C. Zhang, Phys. Rev. Lett. **56**, 889 (1986).
[13] T. Sasamoto and H. Spohn H, Phys. Rev. Lett. **104**, 230602 (2010); G. Amir, I. Corwin, and J. Quastel, Commun. Pure Appl. Math. **64**, 466 (2011).
[14] P. Calabrese and P. Le Doussal, Phys. Rev. Lett. **106**, 250603 (2011).
[15] M. Prähofer and H. Spohn, Phys. Rev. Lett. **84**, 4882 (2000); Physica A **279**, 342 (2000).
[16] T. Kriecherbauer and J. Krug, J. Phys. A: Math. Theor. **43**, 403001 (2010).
[17] I. Corwin, Random Matrices: Theor. Appl. **1**, 1130001 (2012).
[18] S. G. Alves, T. J. Oliveira, and S. C. Ferreira, EPL **96**, 48003 (2011); T. J. Oliveira, S. C. Ferreira, and S. G. Alves, Phys. Rev. E **85**, 010601(R) (2012).
[19] K. A. Takeuchi *et al.*, Sci. Rep. **1**, 34 (2011).
[20] P. Yunker *et al.*, Phys. Rev. Lett. **110**, 035501 (2013).
[21] K.-J. Kim *et al.*, Nature (London) **458**, 740 (2009).
[22] T. Halpin-Healy, Phys. Rev. Lett. **109**, 170602 (2012).
[23] J. Kelling and G. Ódor, Phys. Rev. E **84**, 061150 (2011).
[24] E. A. Eklund *et al.* Phys. Rev. Lett. **67**, 1759 (1991); R. Paniago, R. *et al.* Phys. Rev. B **56**, 13442 (1997); F. Ojeda *et al.*, Phys. Rev. Lett. **84**, 3125 (2000).
[25] R. Cuerno *et al.*, Eur. J. Phys. Special Topics **146**, 427 (2007).
[26] S. Das Sarma, P. Punyindu Chatraphorn, and Z. Toroczkai, Phys. Rev. E **65**, 036144 (2002).
[27] K.-S. Ryu, H. Akinaga, and S.-C. Shin, Nature Phys. **3**, 547 (2007).
[28] M. Nicoli, R. Cuerno, and M. Castro, Phys. Rev. Lett. **102**, 256102 (2009).
[29] C. Misbah, O. Pierre-Louis, and Y. Saito, Rev. Mod. Phys. **82**, 981 (2010).
[30] Y.-P. Zhao *et al.*, Phys. Rev. Lett. **82**, 4882 (1999).
[31] M. Nicoli, M. Castro, and R. Cuerno, J. Stat. Mech.: Theor. Exp. (2009) P02036.
[32] M. Castro *et al.*, New J. Phys. **14**, 103039 (2012).
[33] M. Nicoli, R. Cuerno, and M. Castro, J. Stat. Mech.: Theor. Exp. (2011) P10030.
[34] K. Kassner and C. Misbah, Phys. Rev. E **66**, 026102 (2002).
[35] The real-space representation of $k^\mu h_{\mathbf{k}}$ is proportional [33] to the Cauchy principal value of $\int_{\mathbb{R}^d}[h(\mathbf{r}) - h(\mathbf{r}')]/|\mathbf{r} - \mathbf{r}'|^{d+\mu}\, \mathrm{d}\mathbf{r}'$ for $0 < \mu \leq 2$.
[36] Eq. (2) is a non-equilibrium system with weakly long-range interactions, see D. Mukamel in *Long-Range Interacting Systems*, edited by T. Dauxois, S. Ruffo, and L. F. Cugliandolo (Oxford University Press, Oxford, 2010).
[37] E. Katzav, Phys. Rev. E **68**, 031607 (2003).
[38] See Supplemental Material at XXX for details on the numerical estimation of parameters v_∞ and Γ, and larger versions of Figs. 1 to 3.
[39] A. Edelman and P.-O. Persson, *Numerical Methods for Eigenvalue Distributions of Random Matrices*, arXiv:math-ph/0501068 (2005).
[40] F. Bornemann, Math. Comput. **79**, 871 (2010).
[41] J. Krug and H. Spohn, in *Solids far from equilibrium*, edited by C. Godrèche (Cambridge University Press, Cambridge, England, 1991); N. D. Blair-Stahn, arXiv:math/1005.0649v1 (2010).
[42] M. Nicoli, E. Vivo, and R. Cuerno, Phys. Rev. E **82**, 045202(R) (2010).
[43] T. J. Newman and M. R. Swift, Phys. Rev. Lett. **79**, 2261 (1997).
[44] See e.g. L. Canet *et al.*, Phys. Rev. Lett. **104**, 150601 (2010) and references therein.
[45] G. K. Batchelor, *An Introduction to Fluid Dynamics* (Cambridge University Press, Cambridge, UK, 2000).
[46] R. Hoyle, *Pattern formation: an introduction to methods* (Cambridge University Press, Cambridge, UK, 2006).
[47] M. Castro *et al.*, New J. Phys. **9**, 102 (2007).
[48] S. T. Bramwell *et al.*, Phys. Rev. Lett. **84**, 3744 (2000).
[49] T. J. Oliveira, S. G. Alves, and S. C. Ferreira, arXiv:cond-mat/1302.3750v1 (2013).
[50] M. Castellana *et al.*, Phys. Rev. Lett. **107**, 275701 (2011).